

HILBERTIAN INTERPOLATION

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ABSTRACT. I want to prove that all classical techniques of interpolation and approximation as Lagrange, Taylor, Hermite interpolations Bezier's interpolants, Quasi interpolants, Box splines and others (radial splines, simplicial splines) are derived from a **unique** simple hilbertian scheme. For sake of simplicity, we shall consider only elementary examples which could be easily generalized.

1. HILBERT SPACES.

In the following, we say that a prehilbert space is a vectorial space \mathcal{H} with a scalar product $\langle \cdot | \cdot \rangle$ which is a bilinear and positive mapping from $\mathcal{H} \times \mathcal{H}$ into \mathbb{R} .

Then, we set $\|\cdot\| = (\langle \cdot | \cdot \rangle)^{\frac{1}{2}}$; $\|\cdot\|$ is a norm on \mathcal{H} .

When \mathcal{H} with the norm $\|\cdot\|$ is complete, one say that \mathcal{H} is a Hilbert space.

So, we denote a (pre)hilbert space by the formula $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ (or $(\mathcal{H}, \|\cdot\|)$).

Examples :

$$H^m(a, b) = \left\{ f \in C^{m-1}[a, b] ; \forall t \in [a, b] , f^{(m-1)}(t) = f^{(m-1)}(a) + \int_a^t \varphi(s) ds \right. \\ \left. \text{with } \int_a^b |\varphi(s)|^2 ds < +\infty \right\}$$

One can consider many equivalent topological structures on $H^m(a, b)$ relating to different scalar products, such that :

If $f, g \in H^m(a, b)$,

$$(i) \quad \langle f | g \rangle = \int_a^b \sum_{j=0}^m f^{(j)}(t) . g^{(j)}(t) dt$$

$$(ii) \quad \langle f | g \rangle = \sum_{j=0}^{m-1} f^{(j)}(a) . g^{(j)}(a) + \int_a^b f^{(m)}(t) . g^{(m)}(t) dt$$

$$(iii) \quad \langle f | g \rangle = \sum_{j=0}^{m-1} f(\theta_j) . g(\theta_j) + \int_a^b f^{(m)}(t) . g^{(m)}(t) dt \text{ with } a < \theta_0 < \theta_1 < \dots < \theta_m < b$$

2. REPRODUCING HILBERTIAN KERNELS

Let Ω an arbitrary set and \mathbb{R}^Ω the set of mappings from Ω into \mathbb{R} .

We say that :

$$\begin{aligned} \mathcal{H} &\in \mathcal{Hilb}(\mathbb{R}^\Omega) \text{ iff} \\ \mathcal{H} &\text{ is a vectorial subspace of } \mathbb{R}^\Omega \text{ and} \\ \forall t &\in \Omega, \exists C(t) > 0 \text{ such that } \forall f \in \mathcal{H}, |f(t)| \leq C(t) \|f\|_{\mathcal{H}} \end{aligned}$$

Theorem 1. *There exists an isomorphism between $\mathcal{Hilb}(\mathbb{R}^\Omega)$ and $\mathbb{R}_+^{\Omega \times \Omega}$. So, generally, it is equivalent to consider $\mathcal{H} \in \mathcal{Hilb}(\mathbb{R}^\Omega)$ or its hilbertian kernel $H \in \mathbb{R}_+^{\Omega \times \Omega}$ except for numerical applications.*

3. EXAMPLES OF HILBERTIAN KERNELS

3.1. Polynomial kernels. Let $m \in \mathbb{N}$, $t_0 \in \mathbb{R}$ and :

$$\mathcal{P}_m(\mathbb{R}) = \left\{ \begin{array}{l} \text{polynomials } P \text{ on } \mathbb{R} \text{ of degree, } d^\circ P \leq m \\ \text{with the scalar product} \\ \langle P | Q \rangle = \sum_{j=0}^m P^{(j)}(t_0) \cdot Q^{(j)}(t_0) dt \end{array} \right\}$$

Then :

$$\forall t, s \in \mathbb{R}, H(t, s) = \sum_{j=0}^m \frac{(t - t_0)^j}{j!} \frac{(s - t_0)^j}{j!}$$

3.2. A "Spline" kernel.

$$\mathcal{H} = \left\{ f \in H^1(0, 1) ; f(0) = 0 \text{ and } \forall f, g \in H^1(0, 1), \langle f | g \rangle = \int_0^1 f'(t) \cdot g'(t) dt \right\}$$

Then : $H(t, s) = t - (t - s)_+$.

3.3. Fourier's kernel. Let :

$$\mathcal{H} = \left\{ \begin{array}{l} f \in H^1(0, 2) ; f(0) = f(2), \int_0^1 f(t) dt = 0 \\ \text{and } \forall f, g \in H^1(0, 2), \langle f | g \rangle = \int_0^1 f'(t) \cdot g'(t) dt \end{array} \right\}$$

Then :

$$H(t, s) = \sum_{k=1}^{\infty} \frac{\cos[k\pi(t - s)]}{(k\pi)^2}$$

and :

$$\forall t \in [(0, 2)], \forall f \in \mathcal{H}, f(t) = \langle f | H(\cdot, t) \rangle$$

That is the Fourier's development of f which could be extended periodically.

4. OPERATIONS ON HILBERTIAN SPACES / KERNELS

Let Ω_1 and Ω_2 be arbitrary sets and $\mathcal{H}_j \in \mathcal{Hilb}(\mathbb{R}^{\Omega_j})$ $j = 1, 2$.

We denote by $H_j \in \mathbb{R}_+^{\Omega_j \times \Omega_j}$ the hilbertian kernel of \mathcal{H}_j , $j = 1, 2$, and \perp an operation such that $H = H_1 \perp H_2 \in \mathbb{R}_+^{\Omega \times \Omega}$, where Ω is a convenient set. Then there exists $\mathcal{H} \in \mathbb{R}_+^{\Omega \times \Omega}$, whose the hilbertian kernel is equal to H . So, we set : $\mathcal{H} = \mathcal{H}_1 \perp \mathcal{H}_2$.

Examples :

(i) Let $H_1, H_2 \in \mathbb{R}_+^{\Omega \times \Omega}$.

Then :

$$\lambda H_1 \ (\lambda \geq 0), H_1 + H_2, \sup(H_1, H_2), \inf(H_1, H_2), \dots \in \mathbb{R}_+^{\Omega \times \Omega}$$

(ii) Let $H_j \in \mathbb{R}_+^{\Omega_j \times \Omega_j}$, $j = 1, 2$.

Then, if $\Omega = \Omega_1 \times \Omega_2$:

$$\begin{aligned} H_1 \otimes H_2, \frac{1}{2!} (H_1 \wedge H_2) &= \frac{1}{2!} (H_1 \otimes H_2 + H_1 \otimes H_2), \\ H_1 \vee H_2 &= \frac{1}{2!} (H_1 \otimes H_2 - H_1 \otimes H_2) \in \mathbb{R}_+^{\Omega \times \Omega} \end{aligned}$$

5. EXTENSION

We can consider $Hilb(E)$ when E is a topological vectorial space and $E = \mathcal{C}^0(\Omega)$ or $E = \mathcal{D}'(\Omega)$,

5.1. Interpolation. Let us suppose that :

$\mathcal{H} \in Hilb(\mathbb{R}^\Omega)$, $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, and

(k_0, k_1, \dots, k_n) a free system of vectors in \mathcal{H} .

We denote by Π the following general interpolation problem :

$$\Pi : \inf \{ \|f\| ; f \in \mathcal{H} \text{ and } \langle k_j | f \rangle = \alpha_j, 0 \leq j \leq n \}$$

Then the solution of Π is the element $\sigma = Arg(\Pi)$ such that :

$$\forall t \in \Omega, \sigma(t) = \sum_{j=0}^n \lambda_j \langle k_j | H(\cdot, t) \rangle \text{ with } \lambda_j \in \mathbb{R}, 0 \leq j \leq n$$

Examples :

(i) *Lagrange interpolation.*

Let $\theta_0, \dots, \theta_n \in \mathbb{R}$, $\mathcal{H} = \mathcal{P}_n(\mathbb{R})$, $\forall P, Q \in \mathcal{H}$, $\langle P | Q \rangle = \sum_{j=0}^n P(\theta_j) \cdot Q(\theta_j)$

Then :

$$H(s, t) = \sum_{j=0}^n L_{j,n}(s) \cdot L_{j,n}(t)$$

and :

$$\Pi : \inf \left\{ \sum_{j=0}^n |f(\theta_j)|^2 ; f \in \mathcal{H} \text{ and } f(\theta_k) = \alpha_j, 0 \leq j \leq n \right\}$$

Thus :

$$\sigma = \sum_{j=0}^n \alpha_j L_{j,n} \left(= \sum_{j=0}^n \lambda_j H(\cdot, \theta_j) \right)$$

(ii) *Taylor interpolation (Approximation).*

Let : $t_0 \in \mathbb{R}$ and :

$$\mathcal{H} = \left\{ f ; \forall t \in \mathbb{R}, f(t) = \sum_{j=0}^{\infty} \alpha_j \frac{(t-t_0)^j}{j!} \text{ with } \sum_{j=0}^{\infty} |\alpha_j|^2 < +\infty \right\}$$

(let us remark that : $\alpha_j = f^{(j)}(t_0)$)

We suppose that :

$$\forall f, g \in \mathcal{H}, \langle f | g \rangle = \sum_{j=0}^{\infty} f^{(j)}(t_0) \cdot g^{(j)}(t_0)$$

Then \mathcal{H} with $\langle \cdot | \cdot \rangle$ is a Hilbert space with kernel H such that :

$$\forall s, t \in \mathbb{R} \quad H(s, t) = \sum_{j=0}^{\infty} \frac{(s - t_0)^j}{j!} \frac{(t - t_0)^j}{j!}$$

So :

$$\Pi : \inf \left\{ \sum_{j=0}^{\infty} \left| f^{(j)}(t_0) \right|^2 ; f \in \mathcal{H} \text{ and } f^{(k)}(t_0) = \alpha_k, k \in \mathbb{N} \right\}$$

and :

$$\sigma = \sum_{j=0}^{\infty} \lambda_j \left(\frac{\partial^j}{\partial s^j} H(s, t) \right)_{s=t_0} = \sum_{j=0}^{\infty} \alpha_j \frac{(t - t_0)^j}{j!}$$

(We note that : $\sigma^{(j)}(t_0) = \alpha_j$).

(iii) *Bezier- Bernstein interpolant.*

(*) Let us suppose that :

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{P}_1(\mathbb{R}), \forall P, Q \in \mathcal{P}_1(\mathbb{R}), \langle P | Q \rangle = P(0) \cdot Q(0) + P(1) \cdot Q(1)$$

Then, the hilbrtian kernel of $(\mathcal{H}_j, \langle \cdot | \cdot \rangle)$ is

$$\forall s, t \in \mathbb{R}, H_j(s, t) = st + (1 - s)(1 - t), j = 1, 2.$$

So, as in previous paragraphs, we have :

$$\Pi_j : \inf \left\{ \|P\|_j^2 ; P \in \mathcal{H}_j \text{ and } P(0) = \alpha_0, P(1) = \alpha_1 \right\}$$

and

$$\forall t \in \mathbb{R}, \sigma(t) = \alpha_0(1 - t) + \alpha_1 t$$

(**) *Tensorial product*

Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{P}_1(\mathbb{R} \times \mathbb{R})$

The hilbertian kernel H of \mathcal{H} with its usual scalar product is such that :

$$\forall s, t, s', t' \in \mathbb{R}, H(s, s'; t, t') = H_1(s, t) \cdot H_2(s', t')$$

Moreover, we have :

$$\Pi = \Pi_1 \otimes \Pi_2 : \inf \left\{ \|P\|^2 ; P \in \mathcal{H} \text{ and } P(0, 0) = \alpha_{00}, P(1, 0) = \alpha_{10}, P(0, 1) = \alpha_{01}, P(1, 1) = \alpha_{11} \right\}$$

and :

$$\sigma(s, t) = \alpha_{00}(1 - s)(1 - t) + \alpha_{01}(1 - s)t + \alpha_{10}s(1 - t) + \alpha_{11}st$$

(***) *Restriction to the diagonal*

Now, we consider :

$$\forall s, t \in \mathbb{R}, \tilde{H}(s, t) = H_1(s, t) \cdot H_2(s, t)$$

Then, we can easily verify that \tilde{H} is a hilbertian kernel.

So, $\forall s \in \mathbb{R}, \tau(s) = \sigma(s, s)$ is the solution of the $\tilde{\Pi}$ associated to \tilde{H} .

One can, in the same way study box-splines interpolants.

(iv) *Polynomial spline interpolant.*

Let $m \in \mathbb{N}, m \geq 2$ and $\theta_0, \dots, \theta_{m-1} \in \mathbb{R}, \theta_0 < \dots < \theta_{m-1}$.

Let :

$$\mathcal{H} = \{f \in H^m(a, b) ; f(\theta_j) = 0, j = 0, 1, \dots, (m-1)\}$$

with the scalar product $\langle \cdot | \cdot \rangle$ such that

$$\forall f, g \in H^m(a, b), \langle f | g \rangle = \int_a^b f^{(m)}(t) \cdot g^{(m)}(t) dt$$

If H is the hilbertian kernel of \mathcal{H} we have $\forall s, t \in (a, b)$:

$$H(s, t) = (-1)^m \left[G_m(s, t) - \sum_{j=0}^{m-1} L_{j,m-1}(s) G_m(t, \theta_j) - \sum_{j=0}^{m-1} L_{k,m-1}(t) G_m(\theta_k, s) \right. \\ \left. + \sum_{j,k=0}^{m-1} L_{j,m-1}(s) L_{k,m-1}(t) G_m(\theta_j, \theta_k) \right]$$

$$\text{with } G_m(s, t) = \frac{1}{(2m-1)!} (s-t)_+^{2m-1}.$$

Then we consider :

$$\Pi : \inf \{\|f\| ; f \in \mathcal{H} \text{ and } f(t_j) = \alpha_j, 0 \leq j \leq n \text{ with } t_0 < t_1 < \dots < t_n, n > m\}$$

and :

$$\sigma = \sum_{j=0}^n \lambda_j H(\cdot, t_j) \text{ is called a polynomial spline of odd degree.}$$

6. APPROXIMATION OF THE DIRAC'S FUNCTIONAL

Let $\mathcal{H} = \mathcal{H}^2(0, 1)$ and

$$\forall f, g \in \mathcal{H}, \langle f | g \rangle = f(0)g(0) + f'(0)g'(0) + \int_0^1 f''(t) \cdot g''(t) dt.$$

Let :

$$\mathcal{K} = \mathcal{P}_1(0, 1) \text{ and } \mathcal{L} = \{f \in H^2(0, 1) ; f(0) = f'(0) = 0\}$$

and :

$$\forall f, g \in \mathcal{K}, \langle f | g \rangle_1 = f(0)g(0) + f'(0)g'(0)$$

$$\forall f, g \in \mathcal{L}, \langle f | g \rangle_2 = \int_0^1 f''(t) \cdot g''(t) dt.$$

Then one can prove easily that :

$(\mathcal{H}, \langle \cdot | \cdot \rangle)$ is direct sum of the two Hilbert spaces $(\mathcal{K}, \langle \cdot | \cdot \rangle_1)$ and $(\mathcal{L}, \langle \cdot | \cdot \rangle_2)$.

Let H, K, L the hilbertian kernels of $(\mathcal{H}, \langle \cdot | \cdot \rangle), (\mathcal{K}, \langle \cdot | \cdot \rangle_1)$, and $(\mathcal{L}, \langle \cdot | \cdot \rangle_2)$ (respectively).

So,

$$H = K + L \text{ with:}$$

$$\forall s, t \in (0, 1), K(s, t) = 1 + st, L(s, t) = \frac{(s-t)_+^3}{3!} + \frac{s^2 t}{2!} - \frac{s^3}{3!}$$

Thus :

$$\begin{aligned} \forall f &\in \mathcal{H} , f = f_1 + f_2 , \text{ and :} \\ \forall t &\in (0, 1) , f(t) = \langle f_1 | K(\cdot, t) \rangle_1 + \langle f_2 | L(\cdot, t) \rangle_2 \\ &= f(0) + t f'(0) + \int_0^1 \frac{\partial^2 L}{\partial s^2}(t, s) \cdot f''(s) ds \end{aligned}$$

We remark that : $\int_0^1 \frac{\partial^2 L}{\partial s^2}(t, s) \cdot f''(s) ds$ is the Peano's kernel.

Now, we set :

$$\begin{aligned} E^h f &= f(\cdot + h) , D_h = \frac{1}{h} (E^h - E^0) , D_h^2 = D_h \circ D_h \text{ and :} \\ L_h(s, t) &= D_{h,1}^2 D_{h,2}^2 L(s, t) \end{aligned}$$

Then :

(*) L_h is a B-spline and an hilbertian kernel.

(**) $\lim_{h \rightarrow 0} L_h(\cdot, t) = \delta_t$ in $(\mathcal{C}^0(0, 1))'$

where δ_t is the Dirac's functional.

Let us remark that $\frac{\partial^2 L}{\partial s \partial t}(t, s) = \delta_t(s) (= \delta_s(t))$ in $\mathcal{D}'(0, 1)$ only.

So, we have :

$$\forall t \in (0, 1) , \forall f \in \mathcal{C}^0(0, 1) , \lim_{h \rightarrow 0} \int_0^1 L_h(s, t) f(s) ds = f(t) .$$

References.

- [1] Marc Atteia, Hilbertian kernels and spline functions, North Holland.
 - [2] Marc Atteia et Jean Gaches, Approximation hilbertienne, Splines, Ondelettes, Fractales, EDP Sciences - Grenoble Sciences.
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